FOURTH EDITION

Differential Equations & LINEAR ALGEBRA

Stephen W. Goode • Scott A. Annin

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S. W. Goode dedicates this book to Megan and Tobi

S. A. Annin dedicates this book to Arthur and Juliann, the best parents anyone could ask for

Preface

Like the first three editions of *Differential Equations and Linear Algebra*, this fourth edition is intended for a sophomore level course that covers material in both differential equations and linear algebra. In writing this text we have endeavored to develop the student's appreciation for the power of the general vector space framework in formulating and solving linear problems. The material is accessible to science and engineering students who have completed three semesters of calculus and who bring the maturity of that success with them to this course. This text is written as we would naturally teach, blending an abundance of examples and illustrations, but not at the expense of a deliberate and rigorous treatment. Most results are proven in detail. However, many of these can be skipped in favor of a more problem-solving oriented approach depending on the reader's objectives. Some readers may like to incorporate some form of technology (computer algebra system (CAS) or graphing calculator) and there are several instances in the text where the power of technology is illustrated using the CAS Maple. Furthermore, many exercise sets have problems that require some form of technology for their solution. These problems are designated with a $\diamond$.

In developing the fourth edition we have once more kept maximum flexibility of the material in mind. In so doing, the text can effectively accommodate the different emphases that can be placed in a combined differential equations and linear algebra course, the varying backgrounds of students who enroll in this type of course, and the fact that different institutions have different credit values for such a course. The whole text can be covered in a five credit-hour course. For courses with a lower credit-hour value, some selectivity will have to be exercised. For example, much (or all) of Chapter 1 may be omitted since most students will have seen many of these differential equations topics in an earlier calculus course, and the remainder of the text does not depend on the techniques introduced in this chapter. Alternatively, while one of the major goals of the text is to interweave the material on differential equations with the tools from linear algebra in a symbiotic relationship as much as possible, the core material on linear algebra is given in Chapters 2–7 so that it is possible to use this book for a course that focuses solely on the linear algebra presented in these six chapters. The material on differential equations is contained primarily in Chapters 1 and 8–11, and readers who have already taken a first course in linear algebra can choose to proceed directly to these chapters.

There are other means of eliminating sections to reduce the amount of material to be covered in a course. Section 2.7 contains material that is not required elsewhere in the text, Chapter 3 can be condensed to a single section (Section 3.4) for readers needing only a cursory overview of determinants, and Sections 4.7, 5.4, and the later sections of Chapters 6 and 7 could all be reserved for a second course in linear algebra. In Chapter 8, Sections 8.4, 8.8, and 8.9 can be omitted, and, depending on the goals of the course, Sections 8.5 and 8.6 could either be de-emphasized or omitted completely. Similar remarks apply to Sections 9.7–9.10. At California State University, Fullerton we have a four credit-hour course for sophomores that is based around the material in Chapters 1–9.

Major Changes in the Fourth Edition

Several sections of the text have been modified to improve the clarity of the presentation and to provide new examples that reflect insightful illustrations we have used in our own courses at California State University, Fullerton. Other significant changes within the text are listed below.

1. The chapter on vector spaces in the previous edition has been split into two chapters (Chapters 4 and 5) in the present edition, in order to focus separate attention on vector spaces and inner product spaces. The shorter length of these two chapters is also intended to make each of them less daunting.
2. The chapter on inner product spaces (Chapter 5) includes a new section providing an application of linear algebra to the subject of least squares approximation.
3. The chapter on linear transformations in the previous edition has been split into two chapters (Chapters 6 and 7) in the present edition. Chapter 6 is focused on linear transformations, while Chapter 7 places direct emphasis on the theory of eigenvalues and eigenvectors. Once more, readers should find the shorter chapters covering these topics more approachable and focused.
4. Most exercise sets have been enlarged or rearranged. Over 3,000 problems are now contained within the text, and more than 600 concept-oriented true/false items are also included in the text.
5. Every chapter of the book includes one or more optional projects that allow for more in-depth study and application of the topics found in the text.
6. The back of the book now includes the answer to every True-False Review item contained in the text.

Acknowledgments

We would like to acknowledge the thoughtful input from the following reviewers of the fourth edition: Jamey Bass of City College of San Francisco, Tamar Friedmann of University of Rochester, and Linghai Zhang of Lehigh University.

All of their comments were considered carefully in the preparation of the text.

S.A. Annin: I once more thank my parents, Arthur and Juliann Annin, for their love and encouragement in all of my professional endeavors. I also gratefully acknowledge the many students who have taken this course with me over the years and, in so doing, have enhanced my love for these topics and deeply enriched my career as a professor.

1

First-Order Differential Equations

1.1 Differential Equations Everywhere

A **differential equation** is any equation that involves one or more derivatives of an unknown function. For example,

$$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y^2 = 5\sin x \tag{1.1.1}$$

and

$$\frac{dS}{dt} = e^{3t}(S - 1) \tag{1.1.2}$$

are differential equations. In the differential equation (1.1.1) the unknown function or **dependent variable** is y, and x is the **independent variable**; in the differential equation (1.1.2) the dependent and independent variables are S and t, respectively. Differential equations such as (1.1.1) and (1.1.2) in which the unknown function depends only on a single independent variable are called **ordinary differential equations**. By contrast, the differential equation (*Laplace's equation*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

involves *partial* derivatives of the unknown function $u(x, y)$ of two independent variables x and y. Such differential equations are called **partial differential equations**.

One way in which differential equations can be characterized is by the order of the highest derivative that occurs in the differential equation. This number is called the **order** of the differential equation. Thus, (1.1.1) has order two, whereas (1.1.2) is a first-order differential equation.

The major reason why it is important to study differential equations is that these types of equations pervade all areas of science, technology, engineering, and mathematics. In this section we will illustrate some of the multitude of applications that are described mathematically by differential equations and then, in the remainder of the chapter, introduce several techniques that can be used to study the properties and solutions of differential equations.

Population Models

The *Malthusian model* for the growth of a population of bacteria assumes that the rate at which the culture grows is proportional to the number of bacteria present at that time. If $P(t)$ denotes the number of bacteria in the culture at time t, then this growth model is described mathematically by the first-order differential equation

$$\frac{dP}{dt} = kP, \tag{1.1.3}$$

where k is a constant. Since the culture grows in time, k is positive. Here the unknown function is $P(t)$. In elementary calculus it is shown that all functions that satisfy (1.1.3) are of the form[1]

$$P(t) = Ce^{kt}, \tag{1.1.4}$$

where C is an arbitrary constant. The formula for $P(t)$ given in (1.1.4) is called the **general solution** to the differential equation (1.1.3), since every solution to (1.1.3) can be obtained from (1.1.4) by appropriate choice of C. To determine a **particular solution** to the differential equation, we must be given some extra information that specifies the appropriate value of C corresponding to the solution we require. For example, if P_0 denotes the number of bacteria present at time $t = 0$, then in addition to the differential equation (1.1.3) we also have the **initial condition**

$$P(0) = P_0. \tag{1.1.5}$$

But, according to (1.1.4),

$$P(0) = Ce^{k\cdot 0} = C.$$

Therefore, in order to satisfy the initial condition (1.1.5), we must choose $C = P_0$, in which case the particular solution that is relevant for our problem is

$$P(t) = P_0 e^{kt}.$$

Since k is a positive constant, we see that this model predicts that the bacteria population grows exponentially in time. This is consistent with observations of bacteria populations but does not give an accurate description of the growth of populations in other species (people, insects, fish, aardvarks, ...). More general population models arise under the assumption that the rate of growth of the population at time t is a more general function of P than simply kP. For instance, the **logistic population model** corresponds to the case when we assume that there is a constant birthrate B_0 per individual, and that the death rate per individual is proportional to the instantaneous population. The resulting first-order differential equation is

$$\frac{dP}{dt} = (B_0 - D_0 P)P,$$

[1] Alternatively, this can be derived by writing (1.1.3) as $P^{-1}\frac{dP}{dt} = k$ or, equivalently, $\frac{d(\ln P)}{dt} = k$, which can be integrated directly to yield $\ln P = kt + c$, so that $P(t) = e^{kt} \cdot e^c = Ce^{kt}$, where $C = e^c$.

where B_0 and D_0 are positive constants (the Malthusian model considered previously corresponds to $B_0 = k$, $D_0 = 0$). This differential equation is often written in the equivalent form

$$\frac{dP}{dt} = k\left(1 - \frac{P}{C}\right)P, \tag{1.1.6}$$

where $k = B_0$ and $C = B_0/D_0$. In Section 1.5 we will study the logistic population model in detail, and show that, in contrast to the Malthusian model, it predicts that the population does not increase without bound, but rather approaches a limiting population given by the constant C in Equation (1.1.6). This limiting population is called the **carrying capacity** of the population and represents the maximum population that is sustainable with the given resources. The graph of a typical solution to the differential equation (1.1.6) is given in Figure 1.1.1.

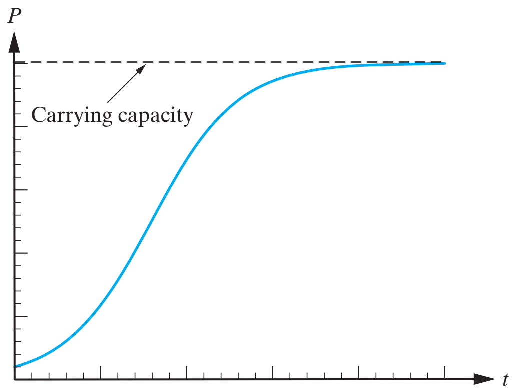

Figure 1.1.1: Behavior of a typical solution to the logistic differential equation (1.1.6).

Newton's Law of Cooling

We now build a mathematical model describing the cooling (or heating) of an object. Suppose that we bring an object into a room. If the temperature of the object is hotter than that of the room, then the object will begin to cool. Further, we might expect that the major factor governing the rate at which the object cools is the temperature difference between it and the room.

Newton's Law of Cooling: The rate of change of temperature of an object is proportional to the difference between the temperature of the object and the temperature of the surrounding medium.

To formulate this law mathematically, we let $T(t)$ denote the temperature of the object at time t, and let $T_m(t)$ denote the temperature of the surrounding medium. Newton's law of cooling can then be expressed as the first-order differential equation

$$\frac{dT}{dt} = -k(T - T_m), \tag{1.1.7}$$

where k is a constant. The minus sign in front of the constant k is traditional. It ensures that k will always be positive.[2] Once we have studied Section 1.4 it will be easy to show that, when T_m is constant, the solution to this differential equation is

$$T(t) = T_m + ce^{-kt}, \tag{1.1.8}$$

[2] If $T > T_m$, then the object will cool, so that $dT/dt < 0$. Hence, from Equation (1.1.7), k must be positive. Similarly, if $T < T_m$, then $dT/dt > 0$, and once more Equation (1.1.7) implies that k must be positive.

where c is a constant (see also Problem 8). Newton's law of cooling therefore predicts that as t approaches infinity ($t \to \infty$) the temperature of the object approaches that of the surrounding medium ($T \to T_m$). This is certainly consistent with our everyday experience (see Figure 1.1.2).

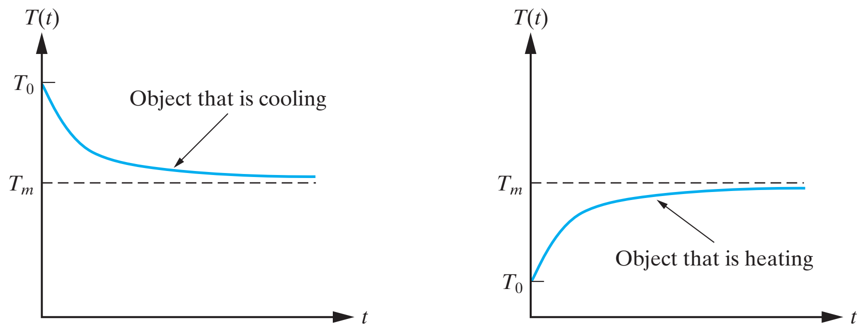

Figure 1.1.2: According to Newton's law of cooling, the temperature of an object approaches room temperature exponentially. In these figures $T_0(= T(0))$ represents the initial temperature of the object.

The Orthogonal Trajectory Problem

Next we consider a geometric problem that has many interesting and important applications. Suppose

$$F(x, y, c) = 0 \tag{1.1.9}$$

defines a family of curves in the xy-plane, where the constant c labels the different curves. For instance if c is a real constant, the equation

$$x^2 + y^2 - c^2 = 0$$

describes a family of concentric circles with center at the origin, whereas

$$-x^2 + y - c = 0$$

describes a family of parabolas that are vertical shifts of the standard parabola $y = x^2$.

We assume that every curve in the family $F(x, y, c) = 0$ has a well-defined tangent line at each point. Associated with this family is a second family of curves, say,

$$G(x, y, k) = 0 \tag{1.1.10}$$

with the property that whenever a curve from the family (1.1.9) intersects a curve from the family (1.1.10) it does so at right angles.[3] We say that the curves in the family (1.1.10) are **orthogonal trajectories** of the family (1.1.9), and vice versa. For example, from elementary geometry, it follows that the lines $y = kx$ in the family $G(x, y, k) = y - kx = 0$ are orthogonal trajectories of the family of concentric circles $x^2 + y^2 = c^2$. (See Figure 1.1.3.)

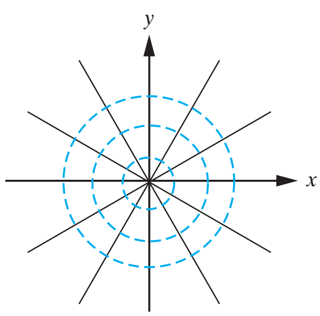

Figure 1.1.3: The family of curves $x^2 + y^2 = c^2$ and the orthogonal trajectories $y = kx$.

Orthogonal trajectories arise in various applications. For example, a family of curves and its orthogonal trajectories can be used to define an orthogonal coordinate system in the xy-plane. In Figure 1.1.3 the families $x^2 + y^2 = c^2$ and $y = kx$ are the coordinate curves of a polar coordinate system (that is, the curves $r =$ constant and $\theta =$ constant,

[3] That is, the tangent lines to each curve are perpendicular at any point of intersection.

respectively). In physics, the lines of electric force of a static configuration are the orthogonal trajectories of the family of equipotential curves. As a final example, if we consider a two-dimensional heated plate, then the heat energy flows along the orthogonal trajectories to the constant temperature curves (isotherms).

Statement of the Problem: Given the equation of a family of curves, find the equation of the family of orthogonal trajectories.

Mathematical Formulation: We recall that curves that intersect at right angles satisfy the following:

The product of the slopes[4] at the point of intersection is -1.

Thus if the given family $F(x, y, c) = 0$ has slope $m_1 = f(x, y)$ at the point (x, y), then the slope of the family of orthogonal trajectories $G(x, y, k) = 0$ at the point (x, y) is $m_2 = -1/f(x, y)$, and therefore the orthogonal trajectories are obtained by solving the first-order differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

Example 1.1.1 Determine the equation of the family of orthogonal trajectories to the curves with equation

$$y^2 = cx. \tag{1.1.11}$$

Solution: According to the preceding discussion, the differential equation determining the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{1}{f(x, y)},$$

where $f(x, y)$ denotes the slope of the given family at the point (x, y). To determine $f(x, y)$, we differentiate Equation (1.1.11) implicitly with respect to x to obtain

$$2y\frac{dy}{dx} = c. \tag{1.1.12}$$

We must now eliminate c from the previous equation to obtain an expression that gives the slope at the point (x, y). From Equation (1.1.11) we have

$$c = \frac{y^2}{x},$$

which, when substituted into Equation (1.1.12), yields

$$\frac{dy}{dx} = \frac{y}{2x}.$$

Consequently, the slope of the given family at the point (x, y) is

$$f(x, y) = \frac{y}{2x}$$

so that the orthogonal trajectories are obtained by solving the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

---

[4] By the slope of a curve at a given point, we mean the slope of the tangent line to the curve at that point.

A key point to notice is that we cannot solve this differential equation by simply integrating with respect to x, since the function on the right-hand side of the differential equation depends on both x and y. However, multiplying by y we see that

$$y\frac{dy}{dx} = -2x$$

or equivalently,

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = -2x.$$

Since the right-hand side of this equation depends only on x whereas the term on the left-hand side is a derivative with respect to x, we can integrate both sides of the equation with respect to x to obtain

$$\frac{1}{2}y^2 = -x^2 + c_1,$$

which we write as

$$2x^2 + y^2 = k \tag{1.1.13}$$

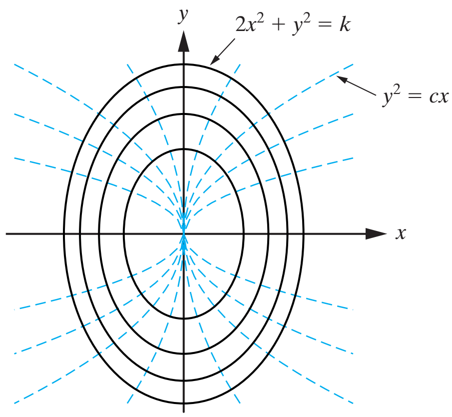

Figure 1.1.4: The family of curves $y^2 = cx$ and its orthogonal trajectories $2x^2 + y^2 = k$.

where $k = 2c_1$. We see that the curves in the given family (1.1.11) are parabolas, and the orthogonal trajectories (1.1.13) are a family of ellipses. This is illustrated in Figure 1.1.4.
□

Newton's Second Law of Motion

Newton's second law of motion states that, for an object of constant mass m, the sum of the applied forces that are acting on the object is equal to the mass of the object multiplied by the acceleration of the object. If the object is moving in one dimension under the influence of a force F, then the mathematical statement of this law is the first-order differential equation

$$m\frac{dv}{dt} = F, \tag{1.1.14}$$

where $v(t)$ denotes the velocity of the object at time t. We let $y(t)$ denote the displacement of the object at time t. Then, using the fact that velocity and displacement are related via

$$v = \frac{dy}{dt}$$

it follows that (1.1.14) can be written as the second-order differential equation

$$m\frac{d^2y}{dt^2} = F. \tag{1.1.15}$$

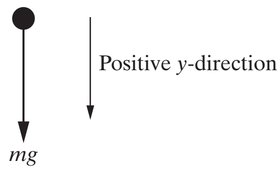

Figure 1.1.5: Object falling under the influence of gravity.

Vertical Motion under Gravity: As a specific example, consider the case of an object falling freely under the influence of gravity (see Figure 1.1.5). In this case the only force acting on the object is $F = mg$, where g denotes the (constant) acceleration due to gravity. It follows from Equation (1.1.15) that the motion of the object is governed by the differential equation[5]

$$m\frac{d^2y}{dt^2} = mg, \tag{1.1.16}$$

or equivalently,

$$\frac{d^2y}{dt^2} = g.$$

Since g is a (positive) constant, we can integrate this equation to determine $y(t)$. Performing one integration yields

$$\frac{dy}{dt} = gt + c_1,$$

where c_1 is an arbitrary integration constant. Integrating once more with respect to t we obtain

$$y(t) = \frac{1}{2}gt^2 + c_1t + c_2, \tag{1.1.17}$$

where c_2 is a second integration constant. We see that the differential equation has an infinite number of solutions parameterized by the constants c_1 and c_2. In order to uniquely specify the motion, we must augment the differential equation with initial conditions that specify the initial position and initial velocity of the object. For example, if the object is released at $t = 0$ from $y = y_0$ with a velocity v_0, then, in addition to the differential equation, we have the initial conditions

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = v_0. \tag{1.1.18}$$

These conditions must be imposed on the solution (1.1.17) in order to determine the values of c_1 and c_2 that correspond to the particular problem under investigation. Setting $t = 0$ in (1.1.17) and using the first initial condition from (1.1.18) we find that

$$y_0 = c_2.$$

Substituting this into Equation (1.1.17), we get

$$y(t) = \frac{1}{2}gt^2 + c_1t + y_0. \tag{1.1.19}$$

In order to impose the second initial condition from (1.1.18), we first differentiate Equation (1.1.19) to obtain

$$\frac{dy}{dt} = gt + c_1.$$

Consequently the second initial condition in (1.1.18) requires

$$c_1 = v_0.$$

[5]Note that we are choosing the positive direction as downward, hence the + sign in front of mg.

From (1.1.19), it follows that the position of the object at time t is

$$y(t) = \frac{1}{2}gt^2 + v_0 t + y_0.$$

The differential equation (1.1.16) together with the initial conditions (1.1.18) is an example of an **initial-value problem**.

A more realistic model of vertical motion under gravity would have to take account of the force due to air resistance. Since increasing velocity generally has the effect of increasing the resistive force, it is reasonable to assume that the force due to air resistance is a function of the instantaneous velocity of the object. A particular model that is often used is to assume that the resistive force is directly proportional to a positive power n (not necessarily integer) of the velocity. Therefore, the total force acting on the object is

$$F = mg - kv^n,$$

where k is a positive constant, so that (1.1.14) can be written as

$$m\frac{dv}{dt} = mg - kv^n. \tag{1.1.20}$$

In Section 1.3 we will develop qualitative techniques for analyzing first-order differential equations that can be used to show that all solutions to Equation (1.1.20) approach a so-called *terminal velocity*, v_T defined by

$$v_T = \lim_{t\to\infty} v(t) = \left(\frac{mg}{k}\right)^{\frac{1}{n}}.$$

This is a very reassuring result for parachutists!

Spring Force: As a second application of Newton's law of motion, consider the spring-mass system depicted in Figure 1.1.6, where, for simplicity, we are neglecting frictional and external forces. In this case, the only force acting on the mass is the restoring force (or spring force), F_s, due to the displacement of the spring from its equilibrium (unstretched) position. We use Hooke's law to model this force:

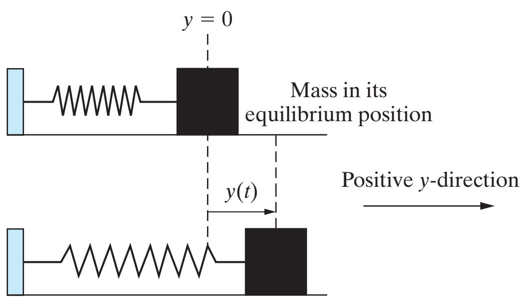

Figure 1.1.6: A simple harmonic oscillator.

Hooke's Law: The restoring force of a spring is directly proportional to the displacement of the spring from its equilibrium position and is directed toward the equilibrium position.

If $y(t)$ denotes the displacement of the spring from its equilibrium position at time t (see Figure 1.1.6), then according to Hooke's law, the restoring force is

$$F_s = -ky,$$

where k is a positive constant called the **spring constant**. Consequently, Newton's second law of motion implies that the motion of the spring-mass system is governed by the differential equation

$$m\frac{d^2y}{dt^2} = -ky$$

which we write in the equivalent form

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \tag{1.1.21}$$

where $\omega = \sqrt{k/m}$. At present we cannot solve this differential equation. However, we leave it as an exercise (Problem 30) to verify by direct substitution that

$$y(t) = A\cos(\omega t - \phi)$$

is a solution to the differential equation (1.1.21), where A and ϕ are constants (determined from the initial conditions for the problem). We see that the resulting motion is periodic with amplitude A. This is consistent with what we might expect physically, since no frictional forces or external forces are acting on the system. This type of motion is referred to as **simple harmonic motion**, and the physical system is called a **simple harmonic oscillator**.

Ontogenetic Growth

Ontogeny is the study of the growth of an individual organism (human, orangutan, snake, fish, ...) from embryo to maximum body size. A general growth equation based purely on fundamental metabolic principles (as opposed to assumptions about birth rates and death rates) has been developed by West, Brown, and Enquist[6] that is applicable to all multicellular animals. The model is derived from the following conservation of energy equation:

$$B(t) = N_c B_c + E_c \frac{dN_c}{dt}, \tag{1.1.22}$$

where $B(t)$ is the resting metabolic rate of the whole organism at time t, B_c is the metabolic rate of a single cell, E_c is the metabolic energy required to create a cell and N_c is the total number of cells. If m and m_c denote the total body mass and average cell mass respectively, then $m = m_c N_c$ so that $N_c = m/m_c$. Substituting this expression for N_c into Equation (1.1.22) yields

$$B = \left(\frac{m}{m_c}\right) B_c + \left(\frac{E_c}{m_c}\right)\frac{dm}{dt}$$

or, equivalently,

$$\frac{dm}{dt} = \left(\frac{m_c}{E_c}\right) B - \left(\frac{m}{E_c}\right) B_c. \tag{1.1.23}$$

Assuming the allometric relationship[7]

$$B = B_0 m^{3/4},$$

where B_0 is a constant, yields

$$\frac{dm}{dt} = \left(\frac{m_c}{E_c}\right) B_0 m^{3/4} - \left(\frac{m}{E_c}\right) B_c, \tag{1.1.24}$$

[6] West, G.B., Brown, J.H., and Enquist, B.J. (2001), *Nature* **400**, 467.

[7] Perhaps surprisingly, this simple relationship between resting metabolic rate and total body mass accurately fits the data across species.

which we write in the equivalent form:

$$\frac{dm}{dt} = am^{3/4}\left[1 - \left(\frac{m}{M}\right)^{1/4}\right], \tag{1.1.25}$$

where $a = B_0 m_c/E_c$ and $M = (B_0 m_c/B_c)^4$. Once we have studied Section 1.4 it will be straightforward to derive the following solution to the differential equation (1.1.25):

$$m(t) = M\left\{1 - \left[1 - \left(\frac{m_0}{M}\right)^{1/4}\right]e^{-at/(4M^{1/4})}\right\}^4. \tag{1.1.26}$$

This solution gives the mass of the animal t days after its birth. We note that

$$\lim_{t\to\infty} m(t) = M,$$

which indicates that the model predicts that organisms do not grow indefinitely but reach a maximum body size, which is represented by the constant M.

Exercises for 1.1

Key Terms

Differential equation, Order of a differential equation, Malthusian population model, Logistic population model, Initial conditions, Newton's law of cooling, Orthogonal trajectories, Newton's second law of motion, Hooke's law, Spring constant, Simple harmonic motion, Simple harmonic oscillator, Ontogenetic growth model.

Skills

- Be able to determine the order of a differential equation.
- Given a differential equation, be able to check whether or not a given function $y = f(x)$ is indeed a solution to the differential equation.
- Be able to describe qualitatively how the temperature of an object changes as a function of time according to Newton's law of cooling.
- Be able to find the equation of the orthogonal trajectories to a given family of curves. In simple geometric cases, be prepared to provide rough sketches of some representative orthogonal trajectories.
- Be able to find the distance, velocity, and acceleration functions for an object moving freely under the influence of gravity.
- Be able to determine the motion of an object in a spring-mass system with no frictional or external forces.

True-False Review

For items (a)–(n), decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

(a) A differential equation for a function $y = f(x)$ must contain the first derivative $y' = f'(x)$.

(b) The order of a differential equation is the order of the lowest derivative appearing in the differential equation.

(c) The differential equation $y'' + e^x(y')^3 + y = \sin x$ has order 3.

(d) In the logistic population model the initial population is called the carrying capacity.

(e) The numerical value $y(0)$ accompanying a first-order differential equation for a function $y = f(x)$ is called an initial condition for the differential equation.

(f) If room temperature is 70° F, then an object whose temperature is 100° F at a particular time cools faster at that time than an object whose temperature at that time is 90° F.

(g) According to Newton's law of cooling, the temperature of an object eventually becomes the same as the temperature of the surrounding medium.

(h) A hot cup of coffee that is put into a cold room cools more in the first hour than the second hour.

(i) At a point of intersection of a curve and one of its orthogonal trajectories, the slopes of the two curves are reciprocals of one another.

(j) The family of orthogonal trajectories for a family of parallel lines is another family of parallel lines.

(k) The family of orthogonal trajectories for a family of circles that are centered at the origin is another family of circles centered at the origin.

(l) The relationship between the velocity and the acceleration of an object falling under the influence of gravity can be expressed mathematically as a differential equation.

(m) Hooke's law states that the restoring force of a spring is directly proportional to the displacement of the spring from its equilibrium position and is directed in the direction of the displacement from the equilibrium position.

(n) According to the ontogenetic growth model the resting metabolic rate of an aardvark is proportional to its mass to the power of three-quarters.

Problems

For Problems 1–4 determine the order of the differential equation.

1. $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = e^x.$

2. $\left(\dfrac{dy}{dx}\right)^3 + y^2 = \sin x.$

3. $y'' + xy' + e^x y = y'''.$

4. $\sin(y'') + x^2y' + xy = \ln x.$

5. Verify that, for $t > 0$, $y(t) = \ln t$ is a solution to the differential equation

$$2\left(\frac{dy}{dt}\right)^3 = \frac{d^3y}{dt^3}.$$

6. Verify that $y(x) = x/(x+1)$ is a solution to the differential equation

$$y + \frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{x^3 + 2x^2 - 3}{(1+x)^3}.$$

7. Verify that $y(x) = e^x \sin x$ is a solution to the differential equation

$$2y \cot x - \frac{d^2y}{dx^2} = 0.$$

8. By writing Equation (1.1.7) in the form

$$\frac{1}{T - T_m}\frac{dT}{dt} = -k$$

and using $u^{-1}\dfrac{du}{dt} = \dfrac{d}{dt}(\ln u)$, derive (1.1.8).

9. A glass of water whose temperature is 50°F is taken outside at noon on a day whose temperature is constant at 70°F. If the water's temperature is 55°F at 2 p.m., do you expect the water's temperature to reach 60°F before 4 p.m. or after 4 p.m.? Use Newton's law of cooling to explain your answer.

10. On a cold winter day (10°F), an object is brought outside from a 70°F room. If it takes 40 minutes for the object to cool from 70°F to 30°F, did it take more or less than 20 minutes for the object to reach 50°F ? Use Newton's law of cooling to explain your answer.

For Problems 11–16, find the equation of the orthogonal trajectories to the given family of curves. In each case, sketch some curves from each family.

11. $x^2 + 9y^2 = c.$

12. $y = cx^2.$

13. $y = c/x.$

14. $y = cx^5.$

15. $y = ce^x.$

16. $y^2 = 2x + c.$

For Problems 17–20, m denotes a fixed nonzero constant, and c is the constant distinguishing the different curves in the given family. In each case, find the equation of the orthogonal trajectories.

17. $y = cx^m.$

18. $y = mx + c.$

19. $y^2 = mx + c.$

20. $y^2 + mx^2 = c.$

21. Consider the family of circles $x^2 + y^2 = 2cx$. Show that the differential equation for determining the family of orthogonal trajectories is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}.$$

22. We call a coordinate system (u, v) **orthogonal** if its coordinate curves (the two families of curves $u =$ constant and $v =$ constant) are orthogonal trajectories (for example, a Cartesian coordinate system or a polar coordinate system). Let (u, v) be orthogonal coordinates, where $u = x^2 + 2y^2$, and x and y are Cartesian coordinates. Find the Cartesian equation of the v-coordinate curves, and sketch the (u, v) coordinate system.

23. Any curve with the property that whenever it intersects a curve of a given family it does so at an angle $a \neq \pi/2$ is called an **oblique trajectory** of the given family. (See Figure 1.1.7.) Let m_1 (equal to $\tan a_1$) denote the slope of the required family at the point (x, y), and let m_2 (equal to $\tan a_2$) denote the slope of the given family. Show that

$$m_1 = \frac{m_2 - \tan a}{1 + m_2 \tan a}.$$

[**Hint**: From Figure 1.1.7, $\tan a_1 = \tan(a_2 - a)$]. Thus, the equation of the family of oblique trajectories is obtained by solving

$$\frac{dy}{dx} = \frac{m_2 - \tan a}{1 + m_2 \tan a}.$$

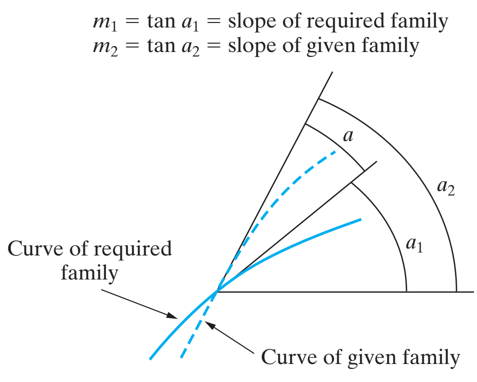

Figure 1.1.7: Oblique trajectories intersect at an angle a.

24. An object is released from rest at a height of 100 meters above the ground. Neglecting frictional forces, the subsequent motion is governed by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 0,$$

where $y(t)$ denotes the displacement of the object from its initial position at time t. Solve this initial-value problem and use your solution to determine the time when the object hits the ground.

25. A five-foot-tall boy tosses a tennis ball straight up from the level of the top of his head. Neglecting frictional forces, the subsequent motion is governed by the differential equation

$$\frac{d^2y}{dt^2} = g.$$

If the object hits the ground 8 seconds after the boy releases it, find

(a) the time when the tennis ball reaches its maximum height.

(b) the maximum height of the tennis ball.

26. A pyrotechnic rocket is to be launched vertically upwards from the ground. For optimal viewing, the rocket should reach a maximum height of 90 meters above the ground. Ignore frictional forces.

(a) How fast must the rocket be launched in order to achieve optimal viewing?

(b) Assuming the rocket is launched with the speed determined in part (a), how long after the rocket is launched will it reach its maximum height?

27. Repeat Problem 26 under the assumption that the rocket is launched from a platform five meters above the ground.

28. An object that is initially thrown vertically upward with a speed of 2 meters/second from a height of h meters takes 10 seconds to reach the ground. Set up and solve the initial-value problem that governs the motion of the object, and determine h.

29. An object that is released from a height h meters above the ground with a vertical velocity of v_0 meters/second hits the ground after t_0 seconds. Neglecting frictional forces, set up and solve the initial-value problem governing the motion, and use your solution to show that

$$v_0 = \frac{1}{2t_0}(2h - gt_0^2).$$

30. Verify that $y(t) = A\cos(\omega t - \phi)$ is a solution to the differential equation (1.1.21), where A and ω are nonzero constants. Determine the constants A and ϕ (with $|\phi| < \pi$ radians) in the particular case when the initial conditions are

$$y(0) = a, \qquad \frac{dy}{dt}(0) = 0.$$

31. Verify that

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

is a solution to the differential equation (1.1.21). Show that the amplitude of the motion is

$$A = \sqrt{c_1^2 + c_2^2}.$$

32. A heron has a birth mass of 3 g, and when fully grown its mass is 2700 g. Using equation (1.1.26) with $a = 1.5$ determine the mass of the heron after 30 days.

33. A rat has a birth mass of 8 g, and when fully grown its mass is 280 g. Using equation (1.1.26) with $a = 0.25$ determine how many days it will take for the rat to reach 75% of its fully grown size.

1.2 Basic Ideas and Terminology

In the previous section we gave several examples of problems that are described mathematically by differential equations. We now formalize many of the ideas introduced through those examples.

Any differential equation of order n can be written in the form

$$G(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{1.2.1}$$

where we have introduced the prime notation to denote derivatives, and $y^{(n)}$ denotes the nth derivative of y with respect to x (not y to the power of n). Of particular interest to us throughout the text will be *linear* differential equations. These arise as the special case of Equation (1.2.1) when $y, y', \ldots, y^{(n)}$ occur to the first degree only, and not as products or arguments of other functions. The general form for such a differential equation is given in the next definition.

DEFINITION 1.2.1

A differential equation that can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n$ and F are functions of x only, is called a **linear** differential equation of order n. Such a differential equation is linear in $y, y', y'', \ldots, y^{(n)}$.

A differential equation that does not satisfy this definition is called a **nonlinear** differential equation.

Example 1.2.2 The equations

$$y''' + e^{3x}y'' + x^3y' + (\cos x)y = \ln x \qquad \text{and} \qquad xy' - \frac{2}{1+x^2}y = 0$$

are linear differential equations of order 3 and order 1, respectively, whereas

$$y'' + x^4\cos(y') - xy = e^{x^2} \qquad \text{and} \qquad y'' + y^2 = 0$$

are both second-order nonlinear differential equations. In the first case the nonlinearity arises from the $\cos(y')$ term, whereas in the second differential equation the nonlinearity is due to the y^2 term. □

Example 1.2.3 The general forms for first- and second-order linear differential equations are

$$a_0(x)\frac{dy}{dx} + a_1(x)y = F(x)$$

and

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x)$$

respectively. □

The differential equation (1.1.3) arising in the Malthusian population model can be written in the form

$$\frac{dP}{dt} - kP = 0$$

and so is a first-order *linear* differential equation. Similarly, writing the Newton's law of cooling differential equation (1.1.7) as

$$\frac{dT}{dt} + kT = kT_m$$

reveals that it also is a first-order *linear* differential equation. In contrast, the logistic differential equation (1.1.6), when written as

$$\frac{dP}{dt} - kP + \left(\frac{k}{C}\right)P^2 = 0,$$

is seen to be a first-order *nonlinear* differential equation. The differential equation (1.1.21) governing the simple harmonic oscillator, namely,

$$\frac{d^2y}{dt^2} + \omega^2 y = 0,$$

is a second-order linear differential equation. In this case the linearity was imposed in the modeling process when we assumed that the restoring force was directly proportional to the displacement from equilibrium (Hooke's law). Not all springs satisfy this relationship. For example, Duffing's Equation

$$m\frac{d^2y}{dt^2} + k_1 y + k_2 y^3 = 0$$

gives a mathematical model of a nonlinear spring-mass system. If $k_2 = 0$, this reduces to the simple harmonic oscillator equation.

Solutions to Differential Equations

We now define precisely what is meant by a solution to a differential equation.

DEFINITION 1.2.4

A function $y = f(x)$ that is (at least) n times differentiable on an interval I is called a **solution** to the differential equation (1.2.1) on I if the substitution $y = f(x)$, $y' = f'(x), \ldots, y^{(n)} = f^{(n)}(x)$ reduces the differential equation (1.2.1) to an identity valid for all x in I. In this case we say that $y = f(x)$ **satisfies** the differential equation.

Example 1.2.5 Verify that for all constants c_1 and c_2, $y(x) = c_1e^{2x} + c_2e^{-2x}$ is a solution to the linear differential equation $y'' - 4y = 0$ for x in the interval $(-\infty, \infty)$.

Solution: The function $y(x)$ is certainly twice differentiable for all real x. Furthermore,

$$y'(x) = 2c_1e^{2x} - 2c_2e^{-2x}$$

and

$$y''(x) = 4c_1e^{2x} + 4c_2e^{-2x} = 4\left(c_1e^{2x} + c_2e^{-2x}\right).$$

Consequently,

$$y'' - 4y = 4\left(c_1e^{2x} + c_2e^{-2x}\right) - 4\left(c_1e^{2x} + c_2e^{-2x}\right) = 0$$

so that $y'' - 4y = 0$ for every x in $(-\infty, \infty)$. It follows from Definition 1.2.4 that the given function is a solution to the differential equation on $(-\infty, \infty)$. □

In the preceding example, x could assume all real values. Often, however, the independent variable will be restricted in some manner. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}(y-1)$$

is undefined when $x \leq 0$ and so any solution would be defined only for $x > 0$. In fact this linear differential equation has solution

$$y(x) = ce^{\sqrt{x}} + 1, \qquad x > 0,$$

where c is a constant. (The reader can check this by plugging in to the given differential equation, as was done in Example 1.2.5. In Section 1.4 we will introduce a technique that will enable us to derive this solution.) We now distinguish two different ways in which solutions to a differential equation can be expressed. Often, as in Example 1.2.5, we will be able to obtain a solution to a differential equation in the explicit form $y = f(x)$, for some function f. However, when dealing with nonlinear differential equations, we usually have to be content with a solution written in implicit form

$$F(x, y) = 0,$$

where the function F defines the solution, $y(x)$, implicitly as a function of x. This is illustrated in Example 1.2.6.

Example 1.2.6

Verify that the relation $x^2 + y^2 - 4 = 0$ defines an implicit solution to the nonlinear differential equation

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Solution: We regard the given relation as defining y as a function of x. Differentiating this relation with respect to x yields[8]

$$2x + 2y\frac{dy}{dx} = 0.$$

That is,

$$\frac{dy}{dx} = -\frac{x}{y}$$

[8]Note that we have used implicit differentiation in obtaining $d(y^2)/dx = 2y \cdot (dy/dx)$.

as required. In this example we can obtain y explicitly in terms of x since $x^2+y^2-4=0$ implies that

$$y = \pm\sqrt{4-x^2}.$$

The implicit relation therefore contains the two explicit solutions

$$y(x) = \sqrt{4-x^2}, \qquad y(x) = -\sqrt{4-x^2},$$

which correspond graphically to the two semi-circles sketched in Figure 1.2.1.

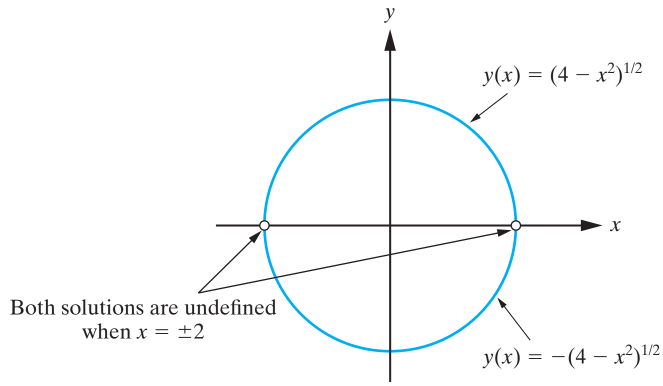

Figure 1.2.1: Two solutions to the differential equation $y' = -x/y$.

Since $x = \pm 2$ correspond to $y = 0$ in both of these equations, whereas the differential equation is only defined for $y \neq 0$, we must omit $x = \pm 2$ from the domains of the solutions. Consequently, both of the foregoing solutions to the differential equation are valid for $-2 < x < 2$. □

In the previous example the solutions to the differential equation are more simply expressed in implicit form although, as we have shown, it is quite easy to obtain the corresponding explicit solutions. In the following example the solution must be expressed in implicit form, since it is impossible to solve the implicit relation (analytically) for y as a function of x.

Example 1.2.7 Verify that the relation $\sin(xy) + y^2 - x = 0$ defines a solution to

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}.$$

Solution: Differentiating the given relationship implicitly with respect to x yields

$$\cos(xy)\left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} - 1 = 0.$$

That is,

$$\frac{dy}{dx}\,[x\cos(xy) + 2y] = 1 - y\cos(xy),$$

which implies that

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy) + 2y}$$

as required. □

Now consider the differential equation

$$\frac{d^2y}{dx^2} = 12x.$$

From elementary calculus we know that all functions whose second derivative is $12x$ can be obtained by performing two integrations. Integrating the given differential equation once yields

$$\frac{dy}{dx} = 6x^2 + c_1,$$

where c_1 is an arbitrary constant. Integrating again we obtain

$$y(x) = 2x^3 + c_1x + c_2, \tag{1.2.2}$$

where c_2 is another arbitrary constant. The point to notice about this solution is that it contains two arbitrary constants. Further, by assigning appropriate values to these constants, we can determine all solutions to the differential equation. We call (1.2.2) the general solution to the differential equation. In this example the given differential equation is of second-order, and the general solution contains two arbitrary constants, which arise due to the fact that two integrations are required to solve the differential equation. In the case of an nth-order differential equation we might suspect that the most general form of solution that can arise would contain n arbitrary constants. This is indeed the case and motivates the following definition.

DEFINITION 1.2.8

A solution to an nth-order differential equation on an interval I is called the **general solution on** I if it satisfies the following conditions:

1. The solution contains n constants $c_1, c_2, \dots, c_n$.
2. All solutions to the differential equation can be obtained by assigning appropriate values to the constants.

Remark Not all differential equations have a general solution. For example, consider

$$(y')^2 + (y-1)^2 = 0.$$

The only solution to this differential equation is $y(x) = 1$, and hence the differential equation does not have a solution containing an arbitrary constant.

Example 1.2.9 Determine the general solution to the differential equation $y'' = 18\cos 3x$.

Solution: Integrating the given differential equation with respect to x yields

$$y' = 6\sin 3x + c_1,$$

where c_1 is an integration constant. Integrating this equation we obtain

$$y(x) = -2\cos 3x + c_1x + c_2, \tag{1.2.3}$$

where c_2 is another integration constant. Consequently, all solutions to $y'' = 18\cos 3x$ are of the form (1.2.3), and therefore, according to Definition 1.2.8, this is the general solution to $y'' = 18\cos 3x$ on any interval. □

As the previous example illustrates, we can, in principle, always find the general solution to a differential equation of the form

$$\frac{d^n y}{dx^n} = f(x) \tag{1.2.4}$$

by performing n integrations. However, if the function on the right-hand side of the differential equation is not a function of x only, this procedure cannot be used. Indeed, one of the major aims of this text is to determine solution techniques for differential equations that are more complicated than Equation (1.2.4).

A solution to a differential equation is called a **particular solution** if it does not contain any arbitrary constants not present in the differential equation itself. One way in which particular solutions arise is by assigning specific values to the arbitrary constants occurring in the general solution to a differential equation. For example, from (1.2.3),

$$y(x) = -2\cos 3x + 5x - 7$$

is a particular solution to the differential equation $d^2y/dx^2 = 18\cos 3x$ (the solution corresponding to $c_1 = 5, c_2 = -7$).

Initial-Value Problems

As discussed in the previous section, the unique specification of an applied problem requires more than just a differential equation. We must also give appropriate auxiliary conditions that characterize the problem under investigation. Of particular interest to us is the case of the initial-value problem defined for an nth-order differential equation as follows.

DEFINITION 1.2.10

An nth-order differential equation together with n auxiliary conditions of the form

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants, is called an **initial-value problem**.

Example 1.2.11 Solve the initial-value problem

$$y'' = 18\cos 3x \tag{1.2.5}$$

$$y(0) = 1, \qquad y'(0) = 4. \tag{1.2.6}$$

Solution: From Example 1.2.9, the general solution to Equation (1.2.5) is

$$y(x) = -2\cos 3x + c_1 x + c_2. \tag{1.2.7}$$

We now impose the auxiliary conditions (1.2.6). Setting $x = 0$ in (1.2.7) we see that

$$y(0) = 1 \qquad \text{if and only if} \qquad 1 = -2 + c_2.$$

So $c_2 = 3$. Using this value for c_2 in (1.2.7) and differentiating the result yields

$$y'(x) = 6\sin 3x + c_1.$$

Consequently

$$y'(0) = 4 \qquad \text{if and only if} \qquad 4 = 0 + c_1$$

and hence $c_1 = 4$. Thus the given auxiliary conditions pick out the particular solution to the differential equation (1.2.5) with $c_1 = 4$, and $c_2 = 3$, so that the initial-value problem has the unique solution

$$y(x) = -2\cos 3x + 4x + 3.$$ □

Initial-value problems play a fundamental role in the theory and applications of differential equations. In the previous example, the initial-value problem had a unique solution. More generally, suppose we have a differential equation that can be written in the **normal** form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

According to Definition 1.2.10, the initial-value problem for such an nth-order differential equation is the following:

Statement of the Initial-Value Problem: Solve

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

subject to

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \dots, y_{n-1}$ are constants.

It can be shown that this initial-value problem always has a unique solution provided f and its partial derivatives with respect to $y, y', \dots, y^{(n-1)}$, are continuous in an appropriate region. This is a fundamental result in the theory of differential equations. In Chapter 8 we will show how the following special case can be used to develop the theory for *linear* differential equations.

Theorem 1.2.12 Let $a_1, a_2, \dots, a_n, F$ be functions that are continuous on an interval I. Then, for any x_0 in I, the initial-value problem

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

has a unique solution on I.

The next example, which we will refer back to on many occasions throughout the remainder of the text, illustrates the power of the preceding theorem.

Example 1.2.13 Prove that the general solution to the differential equation

$$y'' + \omega^2 y = 0, \quad -\infty < x < \infty, \tag{1.2.8}$$

where ω is a nonzero constant, is

$$y(x) = c_1 \cos \omega x + c_2 \sin \omega x, \tag{1.2.9}$$

where c_1, c_2 are arbitrary constants.

Solution: It is a routine computation to verify that $y(x) = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution to the differential equation (1.2.8) on $(-\infty, \infty)$. According to Definition 1.2.8 we must now establish that *every* solution to (1.2.8) is of the form (1.2.9). To that end, suppose that $y = f(x)$ is any solution to (1.2.8). Then according to the preceding theorem, $y = f(x)$ is the *unique* solution to the initial-value problem

$$y'' + \omega^2 y = 0, \quad y(0) = f(0), \quad y'(0) = f'(0). \tag{1.2.10}$$

However, consider the function

$$y(x) = f(0)\cos\omega x + \frac{f'(0)}{\omega}\sin\omega x. \tag{1.2.11}$$

This is of the form $y(x) = c_1\cos\omega x + c_2\sin\omega x$, where $c_1 = f(0)$ and $c_2 = \dfrac{f'(0)}{\omega}$, and therefore solves the differential equation (1.2.8). Further, evaluating (1.2.11) at $x = 0$ yields

$$y(0) = f(0) \quad \text{and} \quad y'(0) = f'(0).$$

Consequently, (1.2.11) solves the initial-value problem (1.2.10). But, by assumption, $y(x) = f(x)$ solves the same initial-value problem. Due to the uniqueness of solution to this initial-value problem it follows that these two solutions must coincide. Therefore,

$$f(x) = f(0)\cos\omega x + \frac{f'(0)}{\omega}\sin\omega x = c_1\cos\omega x + c_2\sin\omega x.$$

Since $f(x)$ was an arbitrary solution to the differential equation (1.2.8) we can conclude that every solution to (1.2.8) is of the form

$$y(x) = c_1\cos\omega x + c_2\sin\omega x$$

and therefore this is the general solution on $(-\infty, \infty)$. □

For the remainder of this chapter, we will focus our attention primarily on first-order differential equations and some of their elementary applications. We will investigate such differential equations qualitatively, analytically, and numerically.

Exercises for 1.2

Key Terms

Linear differential equation, Nonlinear differential equation, General solution to a differential equation, Particular solution to a differential equation, Initial-value problem.

Skills

- Be able to determine whether a given differential equation is linear or nonlinear.
- Be able to determine whether or not a given function $y(x)$ is a particular solution to a given differential equation.
- Be able to determine whether or not a given implicit relation defines a particular solution to a given differential equation.
- Be able to find the general solution to differential equations of the form $y^{(n)} = f(x)$ via n integrations.
- Be able to use initial conditions to find the solution to an initial-value problem.

True-False Review

For items (a)–(e), decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

(a) The general solution to a third-order differential equation must contain three constants.

(b) An initial-value problem always has a unique solution if the functions and partial derivatives involved are continuous.

(c) The general solution to $y'' + y = 0$ is $y(x) = c_1\cos x + 5c_2\cos x$.

(d) The general solution to $y'' + y = 0$ is $y(x) = c_1 \cos x + 5c_1 \sin x$.

(e) The general solution to a differential equation of the form $y^{(n)} = F(x)$ can be obtained by n consecutive integrations of the function $F(x)$.

Problems

For Problems 1–6, determine whether the differential equation is linear or nonlinear.

1. $\dfrac{d^2y}{dx^2} + e^x \dfrac{dy}{dx} = x^2.$

2. $\dfrac{d^3y}{dx^3} + 4\dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} = xy^2 + \tan x.$

3. $yy'' + x(y') - y = 4x \ln x.$

4. $\sin x \cdot y'' + y' - \tan y = \cos x.$

5. $\dfrac{d^4y}{dx^4} + 3\dfrac{d^2y}{dx^2} = x.$

6. $\sqrt{x}y'' + \dfrac{1}{y'} \ln x = 3x^3.$

For Problems 7–21, verify that the given function is a solution to the given differential equation (c_1 and c_2 are arbitrary constants), and state the maximum interval over which the solution is valid.

7. $y(x) = c_1 e^{-5x} + c_2 e^{5x}, \quad y'' - 25y = 0.$

8. $y(x) = c_1 \cos 2x + c_2 \sin 2x, \quad y'' + 4y = 0.$

9. $y(x) = c_1 e^x + c_2 e^{-2x}, \quad y'' + y' - 2y = 0.$

10. $y(x) = \dfrac{1}{x+4}, \quad y' = -y^2.$

11. $y(x) = c_1 x^{1/2}, \quad y' = \dfrac{y}{2x}.$

12. $y(x) = e^{-x} \sin 2x, \quad y'' + 2y' + 5y = 0.$

13. $y(x) = c_1 \cosh 3x + c_2 \sinh 3x, \quad y'' - 9y = 0.$

14. $y(x) = c_1 x^{-3} + c_2 x^{-1}, \quad x^2y'' + 5xy' + 3y = 0.$

15. $y(x) = c_1 x^2 \ln x, \quad x^2y'' - 3xy' + 4y = 0.$

16. $y(x) = c_1 x^2 \cos(3 \ln x), \quad x^2y'' - 3xy' + 13y = 0.$

17. $y(x) = c_1 x^{1/2} + 3x^2, \quad 2x^2y'' - xy' + y = 9x^2.$

18. $y(x) = c_1 x^2 + c_2 x^3 - x^2 \sin x,$
$x^2y'' - 4xy' + 6y = x^4 \sin x.$

19. $y(x) = c_1 e^{ax} + c_2 e^{bx}, \quad y'' - (a+b)y' + aby = 0,$ where a and b are constants and $a \neq b$.

20. $y(x) = e^{ax}(c_1 + c_2 x), \quad y'' - 2ay' + a^2 y = 0,$ where a is a constant.

21. $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx),$
$y'' - 2ay' + (a^2 + b^2)y = 0,$ where a and b are constants.

For Problems 22–25, determine all values of the constant r such that the given function solves the given differential equation.

22. $y(x) = e^{rx}, \quad y'' - y' - 6y = 0.$

23. $y(x) = e^{rx}, \quad y'' + 6y' + 9y = 0.$

24. $y(x) = x^r, \quad x^2y'' + xy' - y = 0.$

25. $y(x) = x^r, \quad x^2y'' + 5xy' + 4y = 0.$

26. When N is a positive integer, the **Legendre equation**

$$(1 - x^2)y'' - 2xy' + N(N+1)y = 0,$$

with $-1 < x < 1$, has a solution that is a polynomial of degree N. Show by substitution into the differential equation that in the case $N = 3$ such a solution is

$$y(x) = \frac{1}{2}x(5x^2 - 3).$$

27. Determine a solution to the differential equation

$$(1 - x^2)y'' - xy' + 4y = 0$$

of the form $y(x) = a_0 + a_1 x + a_2 x^2$ satisfying the normalization condition $y(1) = 1$.

For Problems 28–32, show that the given relation defines an implicit solution to the given differential equation, where c is an arbitrary constant.

28. $x \sin y - e^x = c, \quad y' = \dfrac{e^x - \sin y}{x \cos y}.$

29. $xy^2 + 2y - x = c, \quad y' = \dfrac{1 - y^2}{2(1 + xy)}.$

30. $e^{xy} - x = c, \quad y' = \dfrac{1 - ye^{xy}}{xe^{xy}}.$
Determine the solution with $y(1) = 0$.

31. $e^{y/x} + xy^2 - x = c, \quad y' = \dfrac{x^2(1 - y^2) + ye^{y/x}}{x(e^{y/x} + 2x^2y)}.$